Lecture Notes on Map Projections

ENGO 421: Coordinate Systems and Map Projections
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OBJECTIVES
In these notes we develop a body of theory to understand and analyse the distortion properties of maps that underlie different map projection algorithms. We live in a three-dimensional world on a near-spherical earth. In order to represent this surface into a pictorial form we face the problem of projecting a three-dimensional surface onto a suitable two-dimensional plane. The nature of this projection/transformation depends on how we wish to use the map. For example, navigation, analysis of relationships between phenomena that occur at discrete points or comparisons of different areas on the map. In a map projection, the geometric relationships between features on the earth’s surface will become distorted. We need to minimise the effects of distortion on a particular item, depending on the intended usage of a map.

THE PROCESS OF PROJECTING THE EARTH’S SURFACE

Spherical Approximation
The earth is best represented by an oblate ellipsoid, but for the purposes of these lectures we will consider the earth to be spherical in shape for mathematical simplicity (radius =6371 km). The surface of the sphere is our reference surface and is the surface from which we desire to map.

Generating Globe
The earth is first scaled down before the projection is applied. This scaled down representation is the same as the conventional globe, with which most people will be familiar. The scale factor used is called the Principal Scale and is denoted by $\mu_0$. This is defined as:

$$\mu_0 = \frac{R}{R_E}$$

where $R$ is the radius of the earth and $R_E$ is the radius of the generating globe. All other scale factors between the generating globe and the projection are called particular scale factors.

Map Projection
A one-to-one relationship between points on the surface of the generating globe and on the two-dimensional map is established for most of the points. This one-to-one relationship is never satisfied over all points of the earth since the earth is a continuous surface and the map is bounded. These points where a one-to-one correspondence is not retained are called singular points. Examples are when the poles are represented by lines
on the map, some areas may not be shown at all, or where there are interruptions in the map.

The basic equations to project coordinates on a sphere \((\phi, \lambda)\) onto a map \((x,y)\) are:

\[
x = f(\phi, \lambda) \\
y = g(\phi, \lambda)
\]

where \(J(\phi, \lambda)\) is the \textit{Jacobian condition}

\[
J(\phi, \lambda) = \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \lambda} - \frac{\partial y}{\partial \phi} \frac{\partial x}{\partial \lambda} \neq 0
\]

The process of representing a part of the surface of the earth as a picture on the plane is represented in figure 1 below.

**Figure 1: The Mapping Process**

There are many different types of map projections, but ALL introduce some distortions. The principal scale can be retained along certain lines or at certain points on the map. These are then known as \textit{lines or points of zero distortion}. If one visualises the projection surface as a piece of paper which touches the generating globe at a certain point or along certain lines, then these lines or points where the two surfaces touch will be distortion-free.
Map Projections: Fundamental Development Surfaces and Lines/Points of Zero Distortion

**Plane or Gnomonic Projection**

**Cylindrical Projection**

**Conical Projection**

**References**
Maling, DH, Co-ordinate Systems and Map Projections, Pergamon
Map Projection Basics http://www.geo.hunter.cuny.edu/mp/mpbasics.html
GAUSSIAN DISTORTION THEORY OF MAP PROJECTIONS

We need elements to determine scale and angular distortion on our map projection. One of the most important criteria to consider when choosing a suitable map projection is to minimise the relevant distortions (e.g. scale, angle or area) according to the intended use of the map. In order to do this one needs to consider how these deformations are distributed over the map area. The mathematical tools of map projections provide us with the necessary tools to study the deformation characteristics of a projection in a general sense. The fundamentals of the theory were first laid down by Gauss. This was later extended by the French mathematician, Tissot. Ultimately what we need is to examine a small unit circle on the generating globe and examine the shape of a sample of such circles at various points projected onto the plane. These circles tend to be projected in the shape of an ellipse. We are going to use Gauss’s theory as a means to achieve this.

**Key elements of our ellipse of distortion are:**

- Gaussian fundamental quantities; scale distortion on meridians and parallels, maximum and minimum scale at a point;
- directions of lines on a globe as projected on a plane, and the azimuth and directions of maximum and minimum scale (to find the orientation of our ellipse of distortion) – the principal directions.

We require a simple set of equations and algorithms that we can use to design and analyse the different qualities of particular map projections. In this section we formulate the Gaussian fundamental quantities. We use these to establish the scale distortion along the meridians and parallels, the magnitude, azimuth and plane direction of maximum and minimum scale distortion at a point and the angular distortion at a point on the globe.

Gauss’s theory applies to the projection of any curved surface on another curved surface. The following diagram shows the line element AC=dS on the surface of the generating globe.

\[ x = f(\phi, \lambda) \quad y = g(\phi, \lambda) \]

Need expressions to relate the globe to the map projection coordinates (x,y).

**On the Globe..**

infinitely small quadrilateral.

\[ \theta_m = \text{angle BAC, the azimuth of line increment AC} \]
\[ \phi = \text{latitude} \]
\[ \lambda = \text{longitude} \]

**Figure 2:** A Quadrilateral on the Generating Globe
If, at A, at a given latitude $\phi$ and longitude $\lambda$, then the latitude and longitude at C is given by $\phi + d\phi$ and $\lambda + d\lambda$. The intersection of the meridians and parallels which pass through A and C form a quadrilateral. The line dS forms an angle $\theta_m$ with the meridian, the *azimuth* of dS, and an angle $\theta_p$ with the parallel. As parallels and meridians intersect at right angles (are orthogonal) the sum of $\theta_m$ and $\theta_p$ is 90°. The linear difference along the parallel through point A is called $dS_p$, and the linear difference in the meridian through A is called $dS_m$.

**First Gaussian Fundamental Quantities**

These are the fundamental elements required to determine the magnitudes of distortion parameters. As we shall see, these are used throughout our discussion on distortions in map projections and it is necessary to be able to derive these.

**On the Globe...**

The length of the meridian arc is $dS_m = Rd\phi$

The length of a parallel circle arc is $dS_p = R\cos\phi d\lambda$

The first assumption is that the quadrilateral is infinitely small. This allows the use of an approximation by Pythagoras' theorem:

$$dS^2 = dS_p^2 + dS_m^2$$

$$dS = \sqrt{dS_p^2 + dS_m^2}$$

$$dS = \sqrt{(R\cos\phi d\lambda)^2 + (Rd\phi)^2}$$

If this quadrilateral on the sphere is mapped onto a plane it will be distorted. Generally the direction and length of the lines as well as the direction between them will be distorted. The points A and C on the sphere are mapped to A' and C' on the plane respectively. The lengths of the parallel and meridian elements are denoted by small symbols: ds. The angle of intersection of the quadrilateral sides at A is given by $\theta'$ (which is the distortion of the sum of $\theta_m$ and $\theta_p$ is 90°). Because an assumption that the quadrilateral is infinitely small has been made, the mapped sides and diagonals can also be considered to be straight lines.
Figure 3: Quadrilateral Projected onto the Plane

\[ ds = \sqrt{dx^2 + dy^2} \]

Now \( x = f(\phi, \lambda) \) and \( y = g(\phi, \lambda) \) by definition. We want to express \( x \) and \( y \) in terms of \( \phi \) and \( \lambda \) to relate objects on the globe to the shape of the same object on the plane.

Repeating our sketch on the Plane in more Detail...

On the projected plane....more detailed view
need \( ds \) in terms of lat. and long to get scale along meridian, parallel and along \( ds \).
Differentiating, we get:

\[ dx = \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \lambda} d\lambda \]

\[ dy = \frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial \lambda} d\lambda \]

(The objective is to write our line increment \( ds \) in terms of latitude and longitude increments, \( d\phi \) and \( d\lambda \))

Substituting:

\[ ds^2 = \left( \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \lambda} d\lambda \right)^2 + \left( \frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial \lambda} d\lambda \right)^2 \]

\[ ds^2 = \left( \frac{\partial x}{\partial \phi} \right)^2 + 2 \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \lambda} d\lambda + \left( \frac{\partial x}{\partial \lambda} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 + 2 \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \lambda} d\lambda + \left( \frac{\partial y}{\partial \lambda} \right)^2 \]

\[ ds^2 = \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \right] d\phi^2 + 2 \left[ \left( \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \lambda} \right) \right] d\phi d\lambda + \left[ \left( \frac{\partial x}{\partial \lambda} \right)^2 + \left( \frac{\partial y}{\partial \lambda} \right)^2 \right] d\lambda^2 \]

\[ ds^2 = Ed\phi^2 + 2F d\phi d\lambda + Gd\lambda^2 \]

where

\[ E = \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \right] \]

\[ F = \left[ \left( \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \lambda} \right) \right] \]

\[ G = \left[ \left( \frac{\partial x}{\partial \lambda} \right)^2 + \left( \frac{\partial y}{\partial \lambda} \right)^2 \right] \]

\( E, F, \) and \( G \) are known as the First Gaussian Fundamental Quantities. They are used to study and evaluate the various distortions that take place in a map projection.
SCALE DISTORTION

By what ratio are lines on the globe enlarged or contracted on the plane? To determine this we need to determine the scale distortion along the meridians and parallels first, then we can use these quantities to determine the scale µ along any arc on the globe. We then use this quantity to determine the magnitudes and plane directions of the maximum and minimum scale distortions at a particular point. This provides us with the distortion characteristics of our map projection.

Scale Distortion along the Meridian (µφ)
Referring to the previous figures, an infinitely small quadrilateral projected from the generating globe to the plane.

\[
\mu_\phi = \frac{A'B'}{AB}
\]

\[
(A'B')^2 = (B'P')^2 + (A'P')^2
\]

\[
(B'P') = \frac{\partial x}{\partial \phi} \, d\phi_{AB}
\]

\[
(A'P') = \frac{\partial y}{\partial \phi} \, d\phi_{AB}
\]

\[
(A'B')^2 = \left( \frac{\partial x}{\partial \phi} \, d\phi_{AB} \right)^2 + \left( \frac{\partial y}{\partial \phi} \, d\phi_{AB} \right)^2
\]

(orthogonal components)

\[
(A'B')^2 = E \, d\phi_{AB}^2
\]

\[
A'B' = \sqrt{E} \, d\phi_{AB}
\]

and

\[
AB = dS_m
\]

therefore

\[
\mu_\phi = \frac{\sqrt{E} \, d\phi_{AB}}{dS_m}
\]

\[
\mu_\phi = \frac{\sqrt{E} \, d\phi_{AB}}{R \, d\phi} = \frac{\sqrt{E}}{R}
\]
Scale Distortion along the Parallel Circle ($\mu_\lambda$)

$$\mu_\lambda = \frac{A'D'}{AD}$$

On the sphere: $AD = dS_p$ (see previous figures)

$$(A'D')^2 = (S'D')^2 + (A'S')^2$$

$$(S'D') = \left(\frac{\partial y}{\partial \lambda} d\lambda_{AD}\right)$$

$$(A'S') = \left(\frac{\partial x}{\partial \lambda} d\lambda_{AD}\right)$$

$$(A'D')^2 = \left(\frac{\partial x}{\partial \lambda} d\lambda_{AD}\right)^2 + \left(\frac{\partial y}{\partial \lambda} d\lambda_{AD}\right)^2$$

$$(A'D')^2 = G d\lambda_{AD}^2$$

$$A'D' = \sqrt{G} d\lambda_{AD}$$

and

$$AD = dS_p$$

therefore

$$\mu_\lambda = \frac{\sqrt{G} d\lambda_{AD}}{dS_p} = \frac{\sqrt{G} d\lambda_{AD}}{R \cos \phi d\lambda_{AD}} = \frac{\sqrt{G}}{R \cos \phi}$$
Scale Distortion along an arbitrary arc through point A (μ)

\[ \mu = \frac{A'C'}{AC} = \frac{ds}{dS} \]

\[ dS = \sqrt{R^2 d\phi^2 + R^2 \cos^2 \phi d\lambda^2} \]

\[ ds = \sqrt{E d\phi^2 + 2F d\phi d\lambda + G d\lambda^2} \]

Now, if \( R = 1 \) for a unit spheroid (generating globe has a normalized unit radius):

\[ \mu = \frac{ds}{dS} = \frac{\sqrt{E d\phi^2 + 2F d\phi d\lambda + G d\lambda^2}}{\sqrt{d\phi^2 + \cos^2 \phi d\lambda^2}} \]

If, at a certain point on the map, \( \mu = 1 \), then the true scale \( S \) in that point equals the nominal scale (\( \mu_0 \)). If \( \mu \neq 1 \) then the projection has a distortion in the line element and the true scale is given by: \( S = \mu_0 \cdot \mu \)

**This principle is very important to understand** - the scale indicated on a map is the nominal scale, and is only applicable at those points, or along those lines where \( \mu = 1 \). Generally, scale is not preserved over the entire map projection, but varies from point to point and is often different in all directions at each point.

These three particular scale factors (\( \mu_\phi \), \( \mu_\lambda \), \( \mu \)) relate an infinitesimal linear distance on the map projection with the corresponding linear distance on the globe.
MAXIMUM AND MINIMUM PARTICULAR SCALES

Objective: Our first objective is to find the azimuths of maximum and minimum scale distortion at a point and the plane directions of these distortions on the projected mapping plane. We can use these to determine the distortion characteristics of a particular set of map projection equations by plotting ellipses of distortion at a select number of points on the globe. The plane directions give us the orientation of these ellipses and the maximum and minimum scale factors give us the magnitude of the scale distortion.

Tissot’s theorem is stated as follows:

Whatever the system of projection there are at every point on one of the surfaces orthogonal directions which on the projection plane the vectors which correspond to them also intersect one another at right angles. If angles are not preserved in a particular projection there are only two such directions.

Our first objective is to find the azimuths of the maximum and minimum scales at a particular point, and then compute the plane directions of these maximum and minimum scales on the plane. This gives us the means to plot our ellipses of distortion at a selected sample of points on the mapping plane.

Since the scale distortion varies in different directions, it is interesting to determine in which directions it reaches its maximum and minimum at a particular point on the map. These directions are the axes of our ellipse of distortion. We now need to derive formulae to determine these.

The azimuth angle, A of a line AC (which is equal to angle $\theta_m$ on our initial sketch of the quadrilateral on the generating globe in figure 2) is given by:
\begin{align*}
\tan A &= \frac{R \cos \phi d\lambda}{Rd\phi} = \frac{\cos \phi d\lambda}{d\phi} \\
\frac{d\phi}{\tan A} &= \frac{\cos \phi d\lambda}{d\phi} \\
\frac{d\lambda}{\cos \phi} &= \frac{\tan A d\phi}{\cos \phi}
\end{align*}

Substituting these into the equation for scale distortion:

\[
\mu = \frac{ds}{dS} = \sqrt{\left(Ed\phi^2 + 2Fd\phi d\lambda + Gd\lambda^2\right)}
\]

\[\sqrt{\left(R^2d\phi^2 + R^2 \cos^2 \phi d\lambda^2\right)}\]

However, this last equation is not useful as we do not know the values of \(d\phi\) and \(d\lambda\). We need to eliminate these from our equation. To do this, we express scale in terms of azimuth and the Gaussian fundamental quantities and differentiate scale with respect to azimuth, \(A\), to obtain our azimuth of maximum and minimum distortion.

Let’s ignore the square root for the meantime. Taking each term separately and computing (note that as a trick we multiply the bottom term by \(\frac{d\phi^2}{d\phi^2}\) repeatedly in all cases);

First Term:

\[
\begin{align*}
\frac{Ed\phi^2}{R^2d\phi^2 + R^2 \cos^2 \phi d\lambda^2} &= \frac{E}{R^2} \left(\frac{d\phi^2}{d\phi^2 \left(1 + \cos^2 \phi d\lambda^2 / d\phi^2\right)}\right) \\
&= \frac{E}{R^2} \left(\frac{1}{1 + \cos^2 \phi d\lambda^2 / d\phi^2}\right) \\
&= \frac{E}{R^2} \left(\frac{1}{1 + \tan^2 A}\right) \\
&= \frac{E}{R^2 \sec^2 A}
\end{align*}
\]

*Note* \(\sec^2 A = 1 + \tan^2 A\) is a standard trigonometrical relationship. To recap, \(\cos \phi d\lambda\) is the length increment of our infinitesimally small quadrilateral along the parallel on the sphere (the radius \(R\) cancels out!). And so \(\tan A = \frac{\cos \phi d\lambda}{d\phi}\). We will use these relationships again in manipulating the second term:
Second Term:

\[
\frac{2Fd\phi d\lambda}{R^2 d\phi^2 + R^2 \cos^2 \phi d\lambda^2} = 2F \left( \frac{d\phi d\lambda}{R^2 (d\phi^2 + \cos^2 \phi d\lambda^2)} \right) = 2F \left( \frac{\tan A}{R^2 (1 + \cos^2 \phi \frac{d\lambda^2}{d\phi^2})} \right) = 2F \left( \frac{\tan A}{R^2 \cos \phi} \right) = 2F \sin A \cos A
\]

Third Term:

\[
\frac{Gd\lambda^2}{R^2 d\phi^2 + R^2 \cos^2 \phi d\lambda^2} = G \left( \frac{d\lambda^2}{R^2 (d\phi^2 + \cos^2 \phi d\lambda^2)} \right) = G \left( \frac{\tan^2 A d\phi^2}{\cos^2 \phi \frac{d\lambda^2}{d\phi^2} (1 + \cos^2 \phi \frac{d\lambda^2}{d\phi^2})} \right) = G \left( \frac{\tan^2 A}{R^2 \cos^2 \phi} \right) = G \sin^2 A
\]

Thus, these terms can now be recombined to find an expression for \( \mu \) in terms of azimuth \( A \):

\[
\mu^2 = \frac{E}{R^2} \cos^2 A + \frac{2F}{R^2 \cos \phi} \sin A \cos A + \frac{G}{R^2 \cos^2 \phi} \sin^2 A
\]

To simplify matters, let \( p = \frac{2F}{R^2 \cos \phi} \); and remembering the expressions for scale along the meridian, \( \mu_\phi \), and scale along the parallel, \( \mu_\lambda \), the above expression can be rewritten as:

\[
\mu^2 = \mu_\phi^2 \cos^2 A + p \sin A \cos A + \mu_\lambda^2 \sin^2 A
\]

We need to find the maximum and minimum scales to plot our ellipse of distortion. Now, to find the extremes of scale at a point, the following expression for the extremum must hold true:

\[
\frac{d(\mu^2)}{dA} = 0 \quad \text{(we assume direction of max/min of } \mu^2 \text{ is same as for } \mu)\]
So, differentiating with respect to azimuth $A$:

First Term:

$$\frac{d}{dA} \left( \mu_{\phi}^{2} \cos^{2} A \right) = -2\mu_{\phi}^{2} \sin A \cos A$$

Second Term: (Note$^{1}$)

$$\frac{d}{dA} \left( p \sin A \cos A \right) = p \cos^{2} A - p \sin^{2} A = p \left( \cos^{2} A - \sin^{2} A \right) = p \cos 2A$$

Third Term: (note$^{2}$)

$$\frac{d}{dA} \left( \mu_{\lambda}^{2} \sin^{2} A \right) = 2\mu_{\lambda}^{2} \sin A \cos A$$

Combining these terms and equating to 0:

$$0 = -2\mu_{\phi}^{2} \sin A \cos A + 2\mu_{\lambda}^{2} \sin A \cos A + p \cos 2A$$

$$0 = -2 \sin A \cos A \left( \mu_{\phi}^{2} - \mu_{\lambda}^{2} \right) + p \cos 2A$$

$$0 = -\sin 2A \left( \mu_{\phi}^{2} - \mu_{\lambda}^{2} \right) + p \cos 2A$$

$$\tan 2A_{M} = \frac{p}{\left( \mu_{\phi}^{2} - \mu_{\lambda}^{2} \right)}$$

This indicates that there are two orthogonal azimuths $A_{M}$ and $A_{M} - 90^\circ$ where maximum and minimum scale distortion occur. These directions are called the principal directions (or more correctly, the principal azimuths as they are directions on the globe).

Plug the values of $A_{M}$ and $A_{M} - 90^\circ$ into $\mu^{2} = \mu_{\phi}^{2} \cos^{2} A + p \sin A \cos A + \mu_{\lambda}^{2} \sin^{2} A$ to get maximum and minimum scales.

Let maximum distortion = $a$ Let minimum distortion = $b$

$^{1}$ Rule of differentiation: $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

$^{2}$ Rule of differentiation: $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$
DIRECTIONS ON THE PROJECTED MAP AT A POINT P ON THE GLOBE

Objectives: We need the plane direction on the map of an infinitesimal segment on the globe. We can then determine the principal directions of our ellipse of distortion on the plane. Furthermore we are interested in the angle between the meridians and the parallels on the projected plane. This gives us an idea of angular distortion.

A straight line on the globe will often be projected as a curve on the map. The direction at a point P on the globe is given by the azimuth or by the angles $\theta_m, \theta_p$ in figure 1. On the plane projection the direction of the curves (e.g. meridians and parallels) at the point, P’, are given by the directions $\alpha_m$ and $\alpha_p$ of the tangents at P’.

First objective: If we have an angle $\theta=90^\circ$ between the meridian and parallel on the globe, what is the distortion in this angle at a point on the plane given by the plane angle $\theta’$? (Note to sketch below³.)

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³ Plane coordinate systems: Remember that in most countries directions $\alpha$ are measured from the x-axis toward the y axis in both left and right handed coordinate systems. In a right handed systems as the one above, directions are measured from the x axis increasing anti-clockwise. In our normal geodetic (LH system) they increase clockwise.)
Directions $\alpha$ On The Plane of a Small Increment $ds$ (projected from $dS$ on the Sphere)

In the figure above for any line increment $A'C'$

$$\tan \alpha = \frac{dy}{dx} = \frac{\frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial \lambda} d\lambda}{\frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \lambda} d\lambda}$$

Substituting the equations for $d\phi$ and $d\lambda$ expressed in terms of azimuth determined earlier into the equation for $\tan \alpha$:

$$\tan \alpha = \frac{\delta y \cos \phi d\lambda + \delta y \tan Ad\phi}{\delta \phi \tan A} + \frac{\delta x \cos \phi d\lambda + \delta x \tan Ad\phi}{\delta \lambda \cos \phi}$$

Remember azimuth of $A'C'$ projected onto the plane is the angle $B'A'C'$. 

Taking out a common term $\frac{d\lambda}{\sin A}$:
\[
\tan \alpha = \left( \frac{\frac{\partial y}{\partial \phi} \cos A \cos \phi + \frac{\partial y}{\partial \lambda} \cos A \cos \phi d\lambda}{\frac{\partial x}{\partial \phi} \cos A \cos \phi + \frac{\partial x}{\partial \lambda} \cos A \cos \phi d\lambda} \right)
\]

but:
\[
\frac{\sin^2 A d\phi}{\cos A \cos \phi d\lambda} = \frac{\sin A \sin A d\phi}{\cos A \cos \phi d\lambda} = \frac{\tan A \sin A d\phi}{\cos A \cos \phi d\lambda} = \frac{d\lambda}{d\phi} \frac{\sin A d\phi}{d\lambda} = \sin A
\]

\[
\tan \alpha = \left( \frac{\frac{\partial y}{\partial \phi} \cos A \cos \phi + \frac{\partial y}{\partial \lambda} \sin A}{\frac{\partial x}{\partial \phi} \cos A \cos \phi + \frac{\partial x}{\partial \lambda} \sin A} \right)
\]

Remember from our earlier discussion of the Gaussian fundamental Quantities that we can determine expressions for \(\frac{\partial y}{\partial \phi}, \frac{\partial y}{\partial \lambda}, \frac{\partial x}{\partial \phi}, \frac{\partial x}{\partial \lambda}\) and calculate values for these terms for particular values of \(\phi\) and \(\lambda\).

**Angular Distortion: Angle between Meridian and Parallels**

We have the direction on the plane of any line segment of azimuth \(A\) on the globe. We now want the angle between the meridian and the parallel expressed in terms of the Gaussian fundamental quantities and the scale distortion along the meridians and the parallels.

Now, the new directions of the meridian and parallel can be found by setting \(A=0^\circ\) and \(A=90^\circ\) respectively:

\[
\tan \alpha_m = \frac{\partial y/\partial \phi}{\partial x/\partial \phi}
\]

\[
\tan \alpha_p = \frac{\partial y/\partial \lambda}{\partial x/\partial \lambda}
\]

We want the magnitude of \(\theta'\) without having to calculate the directions of the meridian and parallel projected onto the plane. So, the angle of intersection between the projected meridians and parallels \(\theta'\) can be calculated from:
$$\tan \theta' = \tan (\alpha_m - \alpha_p)$$

$$\tan \theta' = \tan \alpha_m - \tan \alpha_p \over 1 + \tan \alpha_m \tan \alpha_p$$

$$\tan \theta' = \frac{\partial y / \partial \phi - \partial y / \partial \lambda}{1 + \frac{\partial y / \partial \phi}{\partial x / \partial \phi} \frac{\partial y / \partial \lambda}{\partial x / \partial \lambda}}$$

But now from the First Gaussian Fundamental Quantities:

$$EG = \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \right] \left[ \left( \frac{\partial x}{\partial \lambda} \right)^2 + \left( \frac{\partial y}{\partial \lambda} \right)^2 \right]$$

$$EG = \left( \frac{\partial x}{\partial \phi} \right)^2 \left( \frac{\partial x}{\partial \lambda} \right)^2 + \left( \frac{\partial x}{\partial \phi} \right) \left( \frac{\partial y}{\partial \phi} \right) \left( \frac{\partial y}{\partial \lambda} \right) + \left( \frac{\partial y}{\partial \phi} \right) \left( \frac{\partial y}{\partial \lambda} \right)^2$$

$$F^2 = \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 \left( \frac{\partial x}{\partial \lambda} \right)^2 \right] + \left[ \left( \frac{\partial x}{\partial \phi} \right) \left( \frac{\partial y}{\partial \phi} \right) \right] \left[ \left( \frac{\partial y}{\partial \phi} \right) \left( \frac{\partial y}{\partial \lambda} \right) \right]$$

$$F^2 = \left( \frac{\partial x}{\partial \phi} \right)^2 \left( \frac{\partial x}{\partial \lambda} \right)^2 + 2 \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \lambda} + \left( \frac{\partial y}{\partial \phi} \right)^2 \left( \frac{\partial y}{\partial \lambda} \right)^2$$

$$EG - F^2 = \left( \frac{\partial x}{\partial \phi} \right)^2 \left( \frac{\partial y}{\partial \lambda} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \left( \frac{\partial x}{\partial \lambda} \right)^2 - 2 \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \lambda} \frac{\partial y}{\partial \lambda}$$

$$\sqrt{EG - F^2} = \left( \frac{\partial x}{\partial \lambda} \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \lambda} \right)$$

So now:

$$\tan \theta' = \frac{\sqrt{EG - F^2}}{F}$$

It can similarly be proved that

$$\sin \theta' = \sqrt{ \frac{EG - F^2}{EG} }$$

Solving for \( \cos \theta' \):
Thus the angle made at A, at a latitude of $\phi$, by the graticule intersection of the meridian and the parallel can be expressed in terms of a First Gaussian Fundamental Quantities and the scale along the meridian and along the parallel.

Conformal Projections

If $F$ is set to 0 then $\theta' = 90^\circ$, and it is evident that the projected meridians and parallels also intersect at right angles (although they may still be projected as curves). Further, if $F = 0$ then, from the equation for directions of the maximum and minimum scale distortions:

$$\tan 2A_M = \frac{P}{\mu_\phi^2 - \mu_\lambda^2} = 0 = \frac{2F}{R^2 \cos \phi}$$

$$2A_M = \arctan(0) = 0^\circ / 180^\circ / 360^\circ$$

$$A_M = 0^\circ / 90^\circ / 180^\circ.....$$

Thus the principal directions are along the meridians and parallels. Scale distortion in a conformal projection is the same in all directions i.e. It is independent of azimuth. In this case $F = 0$ and $E = \frac{G}{R^2 \cos^2 \phi}$. In this case $\mu_\phi = \mu_\lambda = \mu$. Length distortion is the same in all directions and therefore there is no angular distortion (hence the name conformal).

---

4 By definition a graticule is the position of the intersection of a meridian and a parallel plotted on a map.
Equal-Area Projections

In an equal area (equivalent) projection, elementary areas are preserved. An elementary surface on the generating globe is given by:

$$dS_m dS_p = R^2 \cos\phi d\phi d\lambda.$$  

The same area of the parallelogram on the map is given by: $$dS_m dS_p \sin\theta'.$$

**Area distortion factor** $\sigma$

The surface distortion is thus given by $^6$:

$$\sigma = \frac{dS_m dS_p \sin\theta'}{dS_m dS_p} = \mu_\phi \mu_\lambda \sin\theta'$$

which, as we shall see later, can also be expressed as: $\sigma = ab$

And we know that $\sin\theta' = \sqrt{\frac{EG - F^2}{EG}}$;

and remembering the formulae for $\mu_\phi$ and $\mu_\lambda$: $^7$

Area scale

$$\sigma = \frac{\sqrt{EG - F^2}}{R^2 \cos\phi}.$$  

This parameter $\sigma$ is defined in the same units as the particular scales; therefore it is known as the *area scale*.

Setting this equal to 1 for an equal area projection: $\sqrt{EG - F^2} = R^2 \cos\phi$.

---

$^6$ see previous footnote for parallelogram area expression.

$^7$ $\mu_\phi = \frac{\sqrt{E}}{R}$ and $\mu_\lambda = \frac{\sqrt{G}}{R \cos\phi}$
TISSOT’S INDICATRIX

Tissot’s theory of distortions states that

*A circle on the datum surface with a centre P and a radius ds may be assumed to be a plane figure within its infinitely small area. This area will remain infinitely small and plane on the projection surface. Generally the circle will be portrayed as a ellipse.*

This ellipse is called *Tissot’s Indicatrix* as it indicates the characteristics of a projection in the direct environment of a point.

The axes of Tissot’s Indicatrix correspond to the two principal directions and the maximum and minimum particular scales, $a$ and $b$, at any point, occur in these directions.

**Proof That The Projected Circle Is An Ellipse**

Notes: In figure, the X axis is directed east-west; Y axis is directed north-south. Remember that capital X letters denote elements on the generating globe, and small letters elements on the projection.

\[
\begin{align*}
\text{Sphere} & & \text{Plane} \\
Y & & y \\
\text{dS} & & \text{ds} \\
\theta & & \phi' \\
\text{X} & & x
\end{align*}
\]

\[
\begin{align*}
\text{on the generating globe} & & \text{on the projection} \\
\text{dY} = dS \cdot \sin \theta &= \sqrt{G} \cdot d\lambda \\
\text{dX} = dS \cdot \cos \theta &= \sqrt{E} \cdot d\phi \\
\text{dy} = ds \cdot \sin \theta' &= \sqrt{g} \cdot d\lambda' \\
\text{dx} = ds \cdot \cos \theta' &= \sqrt{e} \cdot d\phi' \\
\text{d} \phi &= \frac{1}{\sqrt{E}} dS \cdot \cos \theta \\
\text{d} \lambda &= \frac{1}{\sqrt{G}} dS \cdot \sin \theta \\
\therefore dy &= \sqrt{g} \cdot \frac{1}{\sqrt{G}} dS \cdot \sin \theta \\
\text{and: } dx &= \sqrt{e} \cdot \frac{1}{\sqrt{E}} dS \cdot \cos \theta
\end{align*}
\]
\[ \frac{dx^2}{E/e} + \frac{dy^2}{G/g} = dS^2 \]

If \( dS = 1 \) then the elementary circle on the globe has a radius of 1
(remember that capital letters denote elements on the generating globe, and small letters elements on the projection.)

\[ \frac{dx^2}{E/e} + \frac{dy^2}{G/g} = 1 \]
This is an equation of an ellipse.

**Analysis of Deformation Characteristics using Tissot’s Indicatrix**

If we call the semi-major and semi-minor axes of the ellipse \( a \), and \( b \), then these are the directions of maximum and minimum distortion i.e. the principal directions. \( a \) and \( b \) are also thus called the *principal scale factors*.

\[ \frac{x^2}{b} + \frac{y^2}{a} = 1 \]

For convenience we will consider the plane \( x \) and \( y \) axes to be in the principal directions.

**Length Distortion**

\[ \mu_x = ds \cos \theta' \quad \text{on the plane} \]
\[ \mu_x = dS \cos \theta = 1 \quad \text{on the globe} \]
(There is no distortion on the globe)

Remember from previous sections: \( \mu = \frac{ds}{dS} \), so:

\[ a = \left( \frac{ds}{dS} \right)_x = \frac{\mu \cos \theta'}{\cos \theta} \]
\[ b = \left( \frac{ds}{dS} \right)_y = \frac{\mu \sin \theta'}{\sin \theta} \]

Or:
\[ a \cos \theta = \mu \cos \theta' \]
\[ b \sin \theta = \mu \sin \theta' \]
\[ a^2 \cos^2 \theta = \mu^2 \cos^2 \theta' \]
\[ b^2 \sin^2 \theta = \mu^2 \sin^2 \theta' \]

\[ a^2 \cos^2 \theta + b^2 \sin^2 \theta = \mu^2 \left( \cos^2 \theta' + \sin^2 \theta' \right) \]
\[ \mu^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta \]

This formula expresses the length distortion in any direction as a function of the original direction \( \theta \), and the principal scale factors, \( a \) and \( b \).

The angle \( \theta \) indicates the direction of the parallel with respect to the x axis. The direction of the meridian with respect to the x axis is thus

\[ \theta + 90^\circ = \theta + \pi / 2 = \beta \]

The scale distortions along the parallels and meridians (note: not necessarily equal to the maximum and minimum distortions along \( a \) and \( b \)) are thus:

\[ \mu^2_H = a^2 \sin^2 \beta + b^2 \cos^2 \beta \]
\[ \mu^2_\Phi = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha = a^2 \cos^2 \beta + b^2 \sin^2 \beta \]
\[ \mu^2_H + \mu^2_\Phi = a^2 + b^2 \]

This is known as the First Theorem of Appolonius:

The sum of the squares of the two conjugate diameters of an ellipse is constant.

**Angular Distortion 2Ω**

Without derivation:

\[ 2\Omega = 2 \arcsin \frac{a - b}{a + b} \]

where \( 2\Omega \) is the maximum angular distortion. The maximum angular deformation occurs in each of the four quadrants.
Figure: Maximum Angular Distortion

If $2 \Omega = 0$ then no angular distortion occurs and the projection is called conformal. The property of a conformal projection is that $a = b$ and Tissot’s Indicatrix is a circle with equal scale distortion in all directions. This is consistent with the previously derived conditions for conformality, namely that $\mu_\Psi = \mu_\lambda$, and $\theta' = \frac{\pi}{2}$. The area is not preserved and the projected circle increases in size as one moves away from the line of zero distortion.

Areal Distortion ($\sigma$): Second Theorem of Appolonius.

This is found by dividing the projected area by the area of the circle on the globe (radius $=1$):

$$\sigma = \frac{\pi ab}{\pi R^2} = ab$$

When looking at equal area projections earlier, it was found that:

$$\sigma = \mu_\phi \mu_\lambda \sin \theta', \text{ thus } \mu_\phi \mu_\lambda \sin \theta' = ab.$$  

This is called the Second Theorem of Appolonius. When $ab = 1$ then the projection is equal-area or equivalent.

Note: conformality and equivalence are exclusive: $ab = 1$ and $a = b$ cannot occur at the same time.
CLASSIFICATION OF MAP PROJECTIONS

Essential Reading: The Geographer’s Craft: Map Projections by Peter Dana
http://www.colorado.edu/geography/gcraft/notes/mapproj/mapproj_f.html

Introduction

It is practical to classify the many map projections according to their visual and distortion characteristics, and also how they relate to each other. This is used extensively in the process of selecting a suitable map projection for a particular purpose. In order to understand the concepts of classification it is necessary first to look at distortion patterns, the aspect of different projections, the special properties, and the transformations. An isogram is a line of uniform distortion.

Projection classes

Azimuthal Projections

This projection can be visualised as a plane touching the globe at one point of zero distortion. The distortion parameters thus increase from this point of contact. The isolines are concentric circles about this point.

Plane, Azimuthal or Gnomic Projection

Cylindrical Projections

Cylindrical projections can be visualised as a roll of paper (the projection surface which is then unrolled and laid flat) which touches the globe transcribing a great circle which is a line of zero distortion. The distortion parameters increase in a direction perpendicular to the line of zero distortion and away from it. Isograms are thus straight lines parallel to the line of zero distortion. The poles are singular points.
Cylindrical Projection

Conical Projections

This projection can be visualised as a piece of paper in the shape of a cone placed over the generating globe. The line where it touches the globe is a small circle which is the line of zero distortion. When the “paper” is unrolled to lie flat the small circle is represented on the map as a circular arc. Singular points occur at the pole and the antipodal points are shown as circular arcs. Isograms are circular and parallel to the line of zero distortion.
ASPECT OF A MAP PROJECTION

The only difference in a map projection between its different aspects is the pattern of the meridians and parallels. The fundamental properties of class are unaltered. The distortion isograms do not alter as the pattern of the graticule changes, and remain in the same position relative to the line/point of zero distortion. A change in aspect cannot thus be considered a different projection. By careful repositioning of the distortion pattern of the projection with respect to the globe, it is possible to determine the aspect which will minimise the distortion in the area of interest.

Normal Aspect

In the above description of the different classes of map projections, the azimuthal and conical examples were given in their normal aspect. This means that for azimuthal the point of zero distortion is at the pole; for cylindrical the line of zero distortion is in the east-west direction, and for the conical projection, that the apex of the cone is over one of the poles. The normal aspect is the one which gives the simplest depiction of the graticule formed by the meridians and parallels. In all normal aspect projections the grid has the same axes of symmetry as the distortion pattern.

Transverse Aspect

The projection surfaces are orientated 90° different to those in the normal aspect. The origin is thus over the Equator, not the poles. The distortion patterns are thus also rotated by 90°.

Oblique Aspect

The projection surface may also be in an infinite number of other orientations. A special case of an oblique aspect is the skew oblique aspect in which symmetry about the central axis is lost. In oblique aspect projections all meridians and parallels are curved, except for the central meridian which is also the single axis of symmetry.
SPECIAL PROPERTIES OF A MAP PROJECTION

These are properties of a map projection which arise from the mutual relationship between the maximum and minimum particular scales at any point, and which are preserved at all singular points of a map i.e. in spite of the principal scale being preserved only at certain lines/points and in spite of variable particular scales. The three special properties are conformality, equivalence, and equidistance. Some of these have previously been mentioned under section 6.3, but the important differences between them will be summarised here.

Conformality

\( a = b \) at all points on the map

Tissot’s Indicatrix is a circle with equal scale distortion in all directions: \( \mu_\phi = \mu_\lambda \)

the projected circle increases in size as one moves away from the line of zero distortion

area is not preserved

angular deformation is zero: \( 2\Omega = 0 \)

meridians and parallels are orthogonal: \( \theta' = \frac{\pi}{2} \)

retains shape for small areas

uses: military, navigation, surveying, topographic mapping

Equivalence (Equal-area)

no areal distortion takes place: \( \sigma = a \times b = \mu_\phi \mu_\lambda \sin \theta' = 1 \) at all points on the map except the singular points

Tissot’s Indicatrix may be an ellipse of great eccentricity but will have the same area as the corresponding circle on the generating globe.

uses: mapping of statistical data to give the correct visual impression of density

Note: conformality and equivalence are exclusive: \( ab = 1 \) and \( a = b \) cannot occur at the same time.

Equidistance

Scale distortion equals unity. This can only be satisfied in one direction and so one particular scale is made equal to the principal scale throughout the map e.g. \( \mu_\phi = 1 \), or \( \mu_\lambda = 1 \). Usually the principal scale is maintained along the meridians.

an equidistant projection is often a good compromise between a conformal and an equivalent projection.

uses: organisations interested in distances between points e.g. airline companies, military.
MAP PROJECTIONS TYPES

Essential Reading Geographer’s Craft Map Projections
http://www.colorado.edu/geography/gcraft/notes/mapproj/mapproj_f.html

ALBERS EQUAL AREA

A conic projection that distorts scale and distance except along standard parallels. Areas are proportional and directions are true in limited areas. Used in the United States and other large countries with a larger east-west than north-south extent.

Conical projection developed by Albers in 1805. Areas are depicted in the same proportion to their actual size. Meridians represented by straight lines converging at a point (the top of the cone). Parallels are concentric circles that are closer together at higher altitudes. One or two standard parallels are used. Position of these parallels should be midway between the middle and limiting parallels of the map (Maling) Distortion is constant along the other parallels Meridians cut the parallels at right angles The pole is not the centre of a circle but a parallel itself. The centre is the apex of the cone which sits above the pole. USGS US National Atlas uses Albers with standard parallels at 29.5° and 45.5°. Useful in the field of distribution mapping or statistical variables (e.g. population density) Useful for large countries that are mainly east – west in extent (e.g. USA, Canada).
MERCATOR

The Mercator projection has straight meridians and parallels that intersect at right angles. Scale is true at the equator or at two standard parallels equidistant from the equator. The projection is often used for marine navigation because all straight lines on the map are lines of constant azimuth.

Cylindrical projection developed graphically by the Belgian cartographer Mercator in 1569.
Conformal or orthomorphic projection in that relative local angles about every point on the map are shown correctly.
Scale at a point is constant in all directions – ellipses of distortion are circles.
Mainly used for navigation
A sailing route along a particular azimuth is depicted as a straight line – a loxodrome or rhumbline.
Gives correct angular relationships in any region provided they are not too close to the poles (<65°?? – check this)
Meridians are vertical equally spaced straight lines, cut at right angles by horizontal straight parallels
Spacing of parallels increases towards the poles to ensure conformality.
Distance is not preserved, but they are “reasonably true” within 15° of equator if equator is standard parallel.
For world charts, scale is true along the equator if this is the standard parallel or along two parallels equidistant from equator. SA navigation charts use regional standard parallels.
Scale factor at any point is secant of the latitude (Iliffe 2000)
Used for world maps, but polar regions are greatly magnified in area (e.g. Greenland).
Pole is a singular point – it is of infinite size
AZIMUTHAL EQUIDISTANT

Azimuthal equidistant projections are sometimes used to show air-route distances. Distances measured from the center are true. Distortion of other properties increases away from the center point.

Point or gnomic projection
Point of zero distortion is usually one of the poles.
All great circles passing through the point of zero distortion are projected as straight lines
In polar aspect, meridians are straight lines radiating from the pole, parallels are concentric circles.
In other aspects, meridians and parallels are complex curves
All directions or azimuths are correct when measured from the centre of the projection.
All distances are measured at true scale when measured between this centre and any other point on the map.
Areal distortion increases as one moves away from the central point.
Useful for setting out a bearing and distance with respect to a select origin –e.g. bird migration patterns from a particular location. Airlines use it for flight lines. Seismic events. Radio transmission, missile launching.
LAMBERT CONFORMAL CONIC

Area, and shape are distorted away from standard parallels. Directions are true in limited areas. Used for maps of North America.

Conical projection developed by Lambert in 1772. 
In polar aspect, zero distortion along one or two standard parallels
In polar aspect, a conformal projection with meridians being equally spaced lines 
radiating from the poles – the pole is a point, unlike Albers Equal Area projection. Looks similar to azimuthal equidistant but distortion characteristics are different.
In polar aspect, parallels are concentric circles becoming further apart as one moves away from standard parallels (compare with mercator)
Parallels cut meridians at 90° in polar aspect.
Distance measurements have to multiplied by a single scale factor, regardless of the direction of the line.
Distortion ellipses are circles increasing in size away from standard parallel(s).
Used for aeronautical charts.
Suitable for countries having a predominant east – west orientation (e.g. Java).
Lambert Conical and UTM make up about 90% of maps world wide (Iliffe 2000)
**UNIVERSAL TRANSVERSE MERCATOR**

Eastings are measured from the central meridian (with a 500km false easting to ensure positive coordinates). Northings are measured from the equator (with a 10,000km false northing for positions south of the equator.

UTM is a cylindrical projection extending between the poles. Developed by the US Defense Mapping Agency to simplify complications in giving directions, distances and positions in military operations. The world is divided into 60 zones of 6° longitude increasing from the Greenwich antimeridian (\(\lambda=180^\circ\)). Zone 1 comprises the belt between 174° and 180°. Horizontal bands of 8° latitude each extend from 80° South to 84° North in each Zone. These are labelled from C (80° S) to X (84° N). As with cadastral diagrams the letters I and O are skipped to avoid confusion with numbers. Band X spans 12° of latitude. A square grid is superimposed on each zone. It has two standard meridians, with a scale factor of 0.9996 along the central meridian. Standard meridians secants are 1.5° either side of central meridian. This leaves 1.5° between the standard meridians and the edges. Has no negative numbers in plane coordinate system based on Northings and Eastings. There are false origins.
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