Pair of Linear Equations in Two Variables

Linear equation in two variables $x$ and $y$ is of the form $ax + by + c = 0$, where $a$, $b$, and $c$ are real numbers, such that both $a$ and $b$ are not zero.

Example: $6x + 3y = 9$, $2x + 1 = 0$, $6x = 1$, $y = 9$

- A linear equation in two variables has infinitely many solutions.
  - Example: $(0, 9)$, $(\frac{9}{2}, 0)$, $(4, 1)$, $(2, 5)$ are some of the solutions of the equation $2x + y = 9$.
  - $\therefore 2 \times 0 + 9 = 9; \ 2 \times \frac{9}{2} + 0 = 9; \ 2 \times 4 + 1 = 9; \ 2 \times 2 + 5 = 9$

- Equations of $x$-axis and $y$-axis are respectively $y = 0$ and $x = 0$.

- The graph of $x = a$ is a straight line parallel to the $y$-axis, and is at a distance of ‘$a$’ units from the $y$-axis.

- The graph of $y = a$ is a straight line parallel to the $x$-axis, and is at a distance of ‘$a$’ units from the $x$-axis.

- Every point on the graph of a linear equation in two variables is a solution of the linear equation and vice versa.

Example: Consider the linear equation $6x + y = 12 \ldots (1)$

$(1, 6)$ is a solution of $(1)$ $\quad [\text{LHS} = 6 \times 1 + 6 = 6 + 6 = 12 = \text{RHS}]$

But, $(2, 3)$ is not a solution of $(1)$ since $LHS = 6 \times 2 + 3 = 12 + 3 = 15 \neq \text{RHS}$

$\therefore$ Point $(1, 6)$ lies on the line representing the equation $(1)$, whereas point $(2, 3)$ does not lie on the line.

Two linear equations in the same two variables are called a pair of linear equations in two variables.

The general form of a pair of linear equations is $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, where $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$ are real numbers such that $a_1^2 + b_1^2 \neq 0, a_2^2 + b_2^2 \neq 0$

Graphical representation of linear equations:

**Example:** Solve the following system of linear equations graphically.

$x + y + 2 = 0, 2x - 3y + 9 = 0$

Hence, find the area bounded by these two lines and the line $x = 0$

**Solution:**

The given equations are

$x + y + 2 = 0 \quad \ldots (1)$

$2x - 3y + 9 = 0 \quad \ldots (2)$
Table for the equations \( x + y + 2 = 0 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( -2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( -2 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

Table for the equation \( 2x - 3y + 9 = 0 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( -4.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( 3 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

By plotting and joining the points \((0, -2)\) and \((-2, 0)\), the line representing equation (1) is obtained.

By plotting and joining the points \((0, 3)\) and \((-4.5, 0)\), the line representing equation (2) is obtained.

It is seen that the two lines intersect at point \(B\) \((-3, 1)\).

\[ \therefore \text{Solution of the given system of equation is } (-3, 1) \]

Area bound by the two lines and \(x = 0\)

\[ = \text{Area of } \triangle ABC \]
\[ = \frac{1}{2} \times AC \times BL = \frac{1}{2} \times 5 \times 3 \text{ square units} = 7.5 \text{ square units} \]

**Consistent system**
A system of simultaneous linear equations is said to be consistent if it has at least one solution.

**Inconsistent system**
A system of simultaneous linear equations is said to be inconsistent if it has no solution.
A pair of linear equations in two variables can be solved by
1) **Graphical method** or
2) **Algebraic method**

Nature of solution of simultaneous linear equations based on graph:—

Case (i): The lines intersect at a point.
   The point of intersection is the unique solution of the two equations.
   In this case, the pair of equations is **consistent**.

Case (ii): The lines coincide.
   The pair of equations has infinitely many solutions — each point on the line is a solution.
   In this case, the pair of equations is dependent (which is **consistent**).

Case (iii): The lines are parallel.
   The pair of equations has no solution. In this case, the pair of equations is **inconsistent**.

**Example:** Show graphically that the system of equations $2x + 3y = 10$, $2x + 5y = 12$ is consistent.

**Solution:**
The given system of linear equations is

\[ \begin{align*}
2x + 3y &= 10 \quad \text{… (1)} \\
2x + 5y &= 12 \quad \text{… (2)}
\end{align*} \]

Table for equation (1)

<table>
<thead>
<tr>
<th>$x$</th>
<th>5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Table for equations (2)

<table>
<thead>
<tr>
<th>$x$</th>
<th>6</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

By plotting and joining the points (5, 0) and (2, 2), the line representing (1) can be obtained.
Similarly, by plotting and joining the points (6, 0) and (1, 2), the line representing (2) can be obtained.
It is seen that the two lines intersect at point \( \left( \frac{7}{2}, 1 \right) \).

Therefore, the given system of equations is consistent and has a unique solution \( \left( \frac{7}{2}, 1 \right) \).

**Example:** Show graphically that the system of equations \( 3x - 6y + 9 = 0 \), \( 2x - 4y + 6 = 0 \) has infinitely many solutions (that is, inconsistent).

**Solution:**

The given system of equations is

\[
\begin{align*}
3x - 6y + 9 &= 0 \\
2x - 4y + 6 &= 0
\end{align*}
\]  \( \ldots \) (1)  \( \ldots \) (2)

Table for equation (1)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(1)</td>
<td>(4)</td>
</tr>
</tbody>
</table>

Table for equation (2)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(0)</th>
<th>(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(1.5)</td>
<td>(2)</td>
</tr>
</tbody>
</table>
By plotting and joining \((-1, 1), (5, 4)\), the line representing equation (1) can be obtained. 
By plotting and joining \((0, 1.5), (1, 2)\), the line representing equation (2) can be obtained.

It is observed that the graphs of the two equations are coincident. Therefore, the given system of equations has infinitely many solutions.

**Example:** By graphical method, check whether the system of equations \(5x + 2y + 10 = 0, 10x + 4y + 15 = 0\) is consistent.

**Solution:**
The given system of equations is
\[5x + 2y + 10 = 0 \quad \ldots (1)\]
\[10x + 4y + 15 = 0 \quad \ldots (2)\]

Table for equation (1)

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-5</td>
<td>0</td>
</tr>
</tbody>
</table>

Table for equation (2)

<table>
<thead>
<tr>
<th>(x)</th>
<th>-1.5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
<td>-3.75</td>
</tr>
</tbody>
</table>
By plotting and joining \((0, -5)\) and \((-2, 0)\), the line representing equation (1) is obtained.

By plotting and joining \((-1.5, 0)\), \((0, -3.75)\), the line representing equation (2) is obtained.

It is seen that the two lines are parallel.

Therefore, the given system of equations has no solution, that is, the system is inconsistent.

**Nature of solution of simultaneous linear equations based on the coefficients:**

Let \(a_1x + b_1y + c_1 = 0\), \(a_2x + b_2y + c_2 = 0\) be a system of linear equations.

**Case (i)** \[\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\]

In this case, the given system is consistent.

This implies that the system has a unique solution.

**Case (ii)** \[\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}\]

In this case, the given system is inconsistent.

This implies that the system has no solution.

**Case (iii)** \[\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}\]

In this case, the given system is dependent and consistent.

This implies that the system has infinitely many solutions.
Example: Find whether the following pairs of linear equations have unique solutions, no solutions, or infinitely many solutions?

- \(7x + 2y + 8 = 0\)
  \(14x + 4y + 16 = 0\)

- \(2x + 3y - 10 = 0\)
  \(5x - 2y - 6 = 0\)

- \(3x - 8y + 12 = 0\)
  \(6x - 16y + 14 = 0\)

Solution:

- \(7x + 2y + 8 = 0\)
  \(14x + 4y + 16 = 0\)
  Here, \(a_1 = 7, b_1 = 2, c_1 = 8\)
  \(a_2 = 14, b_2 = 4, c_2 = 16\)
  Now,
  \[
  \frac{a_1}{a_2} = \frac{7}{14} = \frac{1}{2}, \quad \frac{b_1}{b_2} = \frac{2}{4} = \frac{1}{2}, \quad \frac{c_1}{c_2} = \frac{8}{16} = \frac{1}{2}
  \]
  \[
  \therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}
  \]
  Therefore, the given system has infinitely many solutions.

- \(2x + 3y - 10 = 0\)
  \(5x - 2y - 6 = 0\)
  Here, \(a_1 = 2, b_1 = 3, c_1 = -10\)
  \(a_2 = 5, b_2 = -2, c_2 = -6\)
  Now,
  \[
  \frac{a_1}{a_2} = \frac{2}{5}, \quad \frac{b_1}{b_2} = \frac{3}{-2}
  \]
  \[
  \therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}
  \]
  Therefore, the given system has a unique solution.

- \(3x - 8y + 12 = 0\)
  \(6x - 16y + 14 = 0\)
Here, $a_1 = 3, b_1 = -8, c_1 = 12$

$a_2 = 6, b_2 = -16, c_2 = 14$

Now,

\[
\frac{a_1}{a_2} = \frac{3}{6} = \frac{1}{2}, \quad \frac{b_1}{b_2} = \frac{-8}{-16} = \frac{1}{2}, \quad \frac{c_1}{c_2} = \frac{12}{14} = \frac{6}{7}
\]

\[
\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}
\]

Therefore, the given system has no solutions.

**Algebraic method for solving simultaneous linear equations**

Simultaneous linear equations can be solved algebraically by the following methods.

- **Substitution method**
- **Elimination method**
- **Cross-multiplication method**

**Substitution method**

Example: Solve the following system of equations by substitution method.

\[
x - 4y + 7 = 0
\]

\[
3x + 2y = 0
\]

**Solution:**

The given equations are

\[
x - 4y + 7 = 0 \quad \ldots (1)
\]

\[
3x + 2y = 0 \quad \ldots (2)
\]

From equation (2), $3x = -2y$

\[
\Rightarrow x = -\frac{2}{3}y
\]

Put $x = -\frac{2}{3}y$ in equation (1)
\[-\frac{2}{3}y - 4y + 7 = 0\]
\[\Rightarrow -\frac{2y - 12y}{3} = -7\]
\[\Rightarrow -14y = -21\]
\[\Rightarrow y = \frac{-21}{-14} = \frac{3}{2}\]
\[\therefore x = -\frac{2}{3}\left(\frac{3}{2}\right) = -1\]

Therefore, the required solution is \(-1, \frac{3}{2}\).

\[\textbf{Elimination method}\]

Solve the following pair of linear equations by elimination method.

\[7x - 2y = 10\]
\[5x + 3y = 6\]

\[\textbf{Solution:}\]
\[7x - 2y = 10 \quad \ldots (1)\]
\[5x + 3y = 6 \quad \ldots (2)\]

Multiplying equation (1) by 5 and equation (2) by 7, we get
\[35x - 10y = 50 \quad \ldots (3)\]
\[35x + 21y = 42 \quad \ldots (4)\]

Subtracting equation (4) from (3), we get
\[-31y = 8 \Rightarrow y = -\frac{8}{31}\]

Now, using equation (1):
\[7x = 10 + 2y\]
\[\Rightarrow x = \frac{1}{7}\left(10 + 2\times \frac{-8}{31}\right) = \frac{42}{31}\]
\[\therefore \text{Required solution is } \left(\frac{42}{31}, -\frac{8}{31}\right)\]

\[\textbf{Cross-multiplication method}\]

The solution of the system of linear equations \(a_1x + b_1y + c_1 = 0, \ a_2x + b_2y + c_2 = 0\) can be determined by the following diagram.
That is,
\[
\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}
\]
\[
\Rightarrow \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad (a_1b_2 - a_2b_1 \neq 0)
\]

Example: Solve the following pair of linear equations by the cross-multiplication method
\[
x - 5y = 14, \quad 4x + 3y = 10
\]

Solution:
\[
x - 5y - 14 = 0
\]
\[
4x + 3y - 10 = 0
\]
\[
\begin{array}{ccc}
-5 & x & -14 \\
3 & 1 & 4 \\
\end{array}
\]
\[
\begin{array}{ccc}
-5 & y & 1 \\
3 & 1 & 5 \\
\end{array}
\]
\[
\frac{x}{(-5) \times (-10) - 3 \times (-14)} = \frac{y}{(-14) \times 4 - (-10) \times 1} = \frac{1}{1 \times 3 - 4 \times (-5)}
\]
\[
\Rightarrow \frac{x}{50 + 42} = \frac{y}{-56 + 10} = \frac{1}{3 + 20}
\]
\[
\Rightarrow \frac{x}{92} = \frac{y}{-46} = \frac{1}{23}
\]
\[
\Rightarrow x = \frac{92}{23} = 4, \quad y = \frac{-46}{23} = -2
\]
\[
\therefore \text{Required solution is } (4, -2).
\]

Equations reducible to a pair of linear equations in two variables

- Some pair of equations which are not linear can be reduced to linear form by suitable substitutions.

Example: Solve the following system of equations
\[
\frac{2}{x-2} - \frac{1}{y-1} = 1 \\
\frac{5}{x-2} - \frac{6}{y-1} = 20
\]

**Solution:**

Let \( \frac{1}{x-2} = u \) and \( \frac{1}{y-1} = v \). Then, the given system of equations reduces to

\[
2u - v = 1 \quad \ldots (1) \\
5u - 6v = 20 \quad \ldots (2)
\]

Multiplying equation (1) by 6 and then subtracting from (2), we get

\[
5u - 6v = 20 \\
12u - 6v = 6 \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad \quad \quad \quad \ quarters, it takes 2h 30 min. If he travels 20 km by cycle and the remaining distance by scooter, it takes 4 min longer. Find the speed of the cycle and scooter.

**Solution:**

Let the speed of the cycle be \( x \) km/h,
And the speed of the scooter be \( y \) km/h

Then, we have:

Time taken to travel 15 km by cycle = \( \frac{15}{x} \) h

Time taken to travel \((115 - 15)\) km by scooter = \( \frac{100}{y} \) h

\[
\therefore \frac{15}{x} + \frac{100}{y} = 2 \text{ h 30 min} = \frac{5}{2} \text{ h}
\]

\[
\Rightarrow \frac{15}{x} + \frac{100}{y} = \frac{5}{2}
\]

\[
\Rightarrow 6 + \frac{40}{y} = 1
\]

Time taken to travel 20 km by cycle = \( \frac{20}{x} \) h

Time taken to travel \((115 - 20)\) km by scooter = \( \frac{95}{y} \) h

\[
\therefore \frac{20}{x} + \frac{95}{y} = 2 \text{ h 34 min} = \left(2 + \frac{34}{60}\right) \text{ h}
\]

\[
\Rightarrow \frac{20}{x} + \frac{95}{y} = \frac{77}{30}
\]

\[
\Rightarrow \frac{600}{x} + \frac{2850}{y} = 77
\]

\[\therefore \text{ We have the following system of equations}\]

\[
6 + \frac{40}{y} = 1
\]

\[
600 + \frac{2850}{y} = 77
\]

Let \( \frac{1}{x} = u, \frac{1}{y} = v \)

Then, we have:

\[6u + 40v = 1 \quad \text{... (1)}\]

\[600u + 2850v = 77 \quad \text{... (2)}\]

Multiplying equation (1) by 100 and then subtracting from (2), we have
\[600u + 2850v = 77\]
\[600u + 4000v = 100\]
\[\begin{align*}
-1150v &= -23 \\
v &= \frac{23}{1150} = \frac{1}{50}
\end{align*}\]

From (1), \(6u = 1 - 40v\)
\[\Rightarrow u = \frac{1}{6} \left( 1 - 40 \times \frac{1}{50} \right) = \frac{1}{6} \times \frac{1}{5} = \frac{1}{30}\]
\[\therefore \frac{1}{x} = \frac{1}{30}, \quad \frac{1}{y} = \frac{1}{50}\]
\[\Rightarrow x = 30, \quad y = 50\]

Thus, the speed of the cycle is 30 km/h and the speed of the scooter is 50 km/h.